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Generalized Weierstrass representation for surfaces in multi-dimensional Riemann spaces

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Abstract

Generalizations of the Weierstrass formulae to surface immersed into $\mathbb{R}^4$, $\mathbb{S}^4$ and into multi-dimensional Riemann spaces are proposed. Integrable deformations of surfaces in these spaces via the modified Veselov–Novikov equation are discussed. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Theory of immersion and deformations of surfaces has been an important part of the classical differential geometry (see e.g. [1–3]). Various methods to describe immersions and different types of deformations have been considered. New results in this field have been discussed, for instance, in [4,5].

Surfaces and their dynamics are key ingredients in a number of phenomena in physics too (see e.g. [6–8]). They are, for instance, surface waves, propagation of flame fronts, growth of crystals, deformation of membranes, dynamics of vortex sheets, many problems of hydrodynamics connected with motion of boundaries between region of differing densities and viscosities. A number of papers have been devoted to a study and application of the integrals over surfaces in gauge field theories, string theory, quantum gravity and statistical physics [6–8]).

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Direct approaches to describe surfaces always have been of great interest. The classical Weierstrass formulae for minimal surfaces immersed in the three-dimensional Euclidean space $\mathbb{R}^3$ is the best known example of such an approach. Only recently [9–11] the Weierstrass formulae have been generalized to the case of generic surfaces in $\mathbb{R}^3$. During the last two years the generalized Weierstrass formulae have been used intensively to study both global properties of surfaces in $\mathbb{R}^3$ and their integrable deformations.

In this paper we present the generalizations of the Weierstrass representation for surfaces immersed into the multi-dimensional Euclidean and Riemann spaces. The cases of the four-dimensional Euclidean space $\mathbb{R}^4$ and space $\mathbb{S}^4$ of constant curvature are considered in detail. The comparison of our Weierstrass formulae for surfaces in $\mathbb{R}^4$ and those of conformal immersion in $\mathbb{R}^4$ is given. The properties of the Willmore functional for the immersion in $\mathbb{R}^4$ and $\mathbb{S}^4$ are studied. The Weierstrass representations for immersion into pseudo-Euclidean spaces with signatures $(+,+,+,-)$ and $(+,+,-,-)$ are presented. Surfaces on Lie groups are discussed too.

Integrable deformations of surfaces via the modified Veselov–Novikov equation are considered. It is shown that the Willmore functional (or Helfrich–Polyakov action) is invariant under such deformations.

The paper is organized as follows. In Section 2 a brief review of the old and generalized Weierstrass formulae for surfaces in $\mathbb{R}^3$ is given. The Weierstrass representation for generic surfaces immersed into $\mathbb{R}^4$ is derived in Section 3. An extension to four-dimensional Riemann spaces, in particular to $\mathbb{S}^4$ and the Minkowski space, is presented in Section 4. The Weierstrass representations for surfaces in multi-dimensional spaces are discussed in Section 5. Integrable deformations are considered in Section 6.

2. The old and generalized Weierstrass formulae for surfaces in $\mathbb{R}^3$

Here for the sake of convenience we will briefly remind the old Weierstrass representation for minimal surfaces in $\mathbb{R}^3$ and will review recent results concerned with its generalization to generic surfaces in $\mathbb{R}^3$.

An original Weierstrass formulae (see e.g. [1–3]) starts with two arbitrary holomorphic functions $\varphi(z)$ and $\chi(z)$ of the complex variables $z, \bar{z} \in \mathbb{C}$. Then one introduces the three quantities $X^i(z, \bar{z})$ $(i = 1, 2, 3)$ as follows:

$$
\begin{aligned}
X^1 &= \operatorname{Re}\left[\mathrm{i} \int_\Gamma (\varphi^2 + \chi^2)\, \mathrm{d}z' \right], \\
X^2 &= \operatorname{Re}\left[\int_\Gamma (\varphi^2 - \chi^2)\, \mathrm{d}z' \right], \\
X^3 &= -\operatorname{Re}\left[2 \int_\Gamma \varphi\chi\, \mathrm{d}z' \right].
\end{aligned}
\tag{2.1}
$$

Finally one treats $X^i(z,\bar{z})$ $(i = 1,2,3)$ as the coordinates of a surface in $\mathbb{R}^3$: this surface is a *minimal* one (i.e. has zero mean curvature) and the parametric lines z = const and $\bar{z}$ = const are the minimal lines. The Weierstrass formulae have been the most powerful tool in the study of minimal surfaces.

An analog of the Weierstrass formulae for surfaces of prescribed (non-zero) mean curvature have been proposed by Kenmotsu in [9]. The Kenmotsu representation is given by

$$X^i = \mathrm{Re}\left[\int^{z} \eta\phi^i \, \mathrm{d}z'\right]. \tag{2.2}$$

where $\phi = [1-f^2, \mathrm{i}(1+f^2), 2f]$ and the functions f and η obey the following compatibility condition

$$(\log\eta)_{\bar{z}} = -\frac{2\bar{f} f_{\bar{z}}}{1+|f|^2}. \tag{2.3}$$

Here and below the bar denotes the complex conjugation. Then the mean curvature H is

$$H = -\frac{2\bar{f}_z}{\eta(1+|f|^2)^2}. \tag{2.4}$$

It was proved in [9] that any surface in $\mathbb{R}^3$ can be represented in such a form. This representation of surfaces deals basically with the Gauss map for generic surface in $\mathbb{R}^3$ (see also [12,13]).

Another generalization of the Weierstrass formulae to generic surfaces in $\mathbb{R}^3$ have been proposed independently by one of the authors in 1993 (see [10,11]). It starts with the linear system (two-dimensional Dirac equation)

$$\psi_z = p\varphi, \quad \varphi_{\bar{z}} = -p\psi, \tag{2.5}$$

where ψ and φ are complex-valued functions of $z,\bar{z} \in \mathbb{C}$ and $p(z,\bar{z})$ is a real-valued function. Then one defines the three real-valued functions $X^1(z,\bar{z})$, $X^2(z,\bar{z})$ and $X^3(z,\bar{z})$ by the formulae [10,11]

$$\begin{aligned}
X^1 + \mathrm{i}X^2 &= \mathrm{i}\int_\Gamma (\bar{\psi}^2 \, \mathrm{d}z' - \bar{\varphi}^2 \, \mathrm{d}\bar{z}'),\\
X^1 - \mathrm{i}X^2 &= \mathrm{i}\int_\Gamma (\varphi^2 \, \mathrm{d}z' - \psi^2 \, \mathrm{d}\bar{z}'),\\
X^3 &= -\int_\Gamma (\bar{\psi}\varphi \, \mathrm{d}z' + \psi\bar{\varphi} \, \mathrm{d}\bar{z}').
\end{aligned} \tag{2.6}$$

where Γ is an arbitrary curve in $\mathbb{C}$. In virtue of (2.5) the rhs in (2.6) do not depend on the choice of Γ. If one now treats $X^i(z,\bar{z})$ as the coordinates in $\mathbb{R}^3$ then formulae (2.5), (2.6) define a conformal immersion of surface into $\mathbb{R}^3$ with the induced metric of the form

$$ds^2 = u^2 \, dz \, d\bar{z}, \quad u = |\psi|^2 + |\varphi|^2, \tag{2.7}$$

with the Gauss curvature

$$K = -\frac{4}{u^2}[\log u]_{z\bar{z}} \tag{2.8}$$

and the mean curvature

$$H = 2\frac{p}{u}. \tag{2.9}$$

At $p = 0$ one gets minimal surfaces and formulae (2.6) are reduced to the old Weierstrass formulae (2.1) under the identification $\varphi = \varphi$, $\chi = \overline{\psi}$.

It turned out that the Kenmotsu formulae (2.2), (2.3) and the generalized Weierstrass formulae (2.5), (2.6) are equivalent to each other. The relation between the functions (f, η) and (ψ, φ) is the following [14]:

$$f = \mathrm{i}\frac{\overline{\psi}}{\varphi}, \quad \eta = \mathrm{i}\varphi^2 \tag{2.10}$$

and

$$p = -\frac{\eta f_{\bar{z}}}{\sqrt{\eta\bar{\eta}}(1 + |f|^2)}. \tag{2.11}$$

So, all results proved for the Kenmotsu formulae [9] and associated Gauss map [13] in $\mathbb{R}^3$ are valid also for the generalized Weierstrass formulae (2.6). In particular, it implies immediately that any surface in $\mathbb{R}^3$ can be represented via (2.5), (2.6).

Though the representations (2.2)–(2.4) and (2.5), (2.6) are equivalent, the latter provides us certain advantages. They are mainly due to the fact that in the generalized Weierstrass formulae the functions ψ and φ obey linear equations (2.5) while for the Kenmotsu formulae the nonlinear constraint (2.3) is hard to treat. This circumstance had allowed to simplify essentially an analysis that had lead to several interesting results both of local and global character [14–21]. It occurred, in particular, that the Willmore functional (see e.g. [22]) or the Helfrich–Polyakov action (see [6–8]) $W = \int \boldsymbol{H}^2 [\,dS]$ has a very simple form: $W = 4\int p^2 \, dx \, dy$ $(z = x + \mathrm{i}y)$ [14,15].

One of the advantages of the generalized Weierstrass formulae (2.5), (2.6) is that they allow to construct a new class of deformations of surfaces via the modified Veselov–Novikov equation [10,11]. The characteristic feature of these integrable deformations is that the Willmore functional remains invariant [14,15]. Thus, the generalized Weierstrass representation (2.6) has been proved to be an effective tool to study surfaces in $\mathbb{R}^3$ and their integrable deformations.

We would like to emphasize that the idea to generate surfaces via solutions of linear equations is, in fact, the old idea of the classical differential geometry (see discussion in [11]). In [3] one can find the two representations of these type in addition to the Weierstrass formulae. The first is given by the Lelieuvre's formula which is well known in affine geometry (see e.g. [23]). Another example [3, p. 82] is provided by the equation

$$\theta_{\xi\eta} - (\log\lambda)_\eta\theta_\xi - \lambda^2\theta = 0, \tag{2.12}$$

where ξ, η are real variables and λ is a real-valued function. It is stated in [3] that two solutions of (2.12) define, via certain integral formulae, a surface in $\mathbb{R}^3$ parametrized by minimal lines, but no calculation of the metric and curvature is given. This example, seems, was forgotten completely until it had been found during the preparation of the second paper [11] on the generalized Weierstrass formulae. The representation (2.12) is rather close to that of (2.5), (2.6). Indeed, Eq. (2.12) can be rewritten as the system

$$\theta_\xi = \lambda\varphi, \quad \varphi_\eta = \lambda\theta, \tag{2.13}$$

where φ is a new function. If one takes two solutions (θ, φ) and $(\tilde{\theta}, \tilde{\varphi})$ of the system (2.13) then the formulae given in [3, p. 82] take the form

$$\begin{aligned} X^1 + \mathrm{i}X^2 &= \int (\theta^2\,\mathrm{d}\eta + \varphi^2\,\mathrm{d}\xi), \\ X^1 - \mathrm{i}X^2 &= \int (\tilde{\theta}^2\,\mathrm{d}\eta + \tilde{\varphi}^2\,\mathrm{d}\xi), \\ X^3 &= \mathrm{i}\int (\theta\tilde{\theta}\,\mathrm{d}\eta + \varphi\tilde{\varphi}\,\mathrm{d}\xi). \end{aligned} \tag{2.14}$$

However, in contrast to the representation (2.5), (2.6), formulae (2.13), (2.15) do not define a real surface in $\mathbb{R}^3$.

We would like to note that some results in [24,25] were close to the generalized Weierstrass representation (2.6). In [24] a formula similar to (2.5) for constant mean curvature surfaces has been discussed. In [25] the system (2.5) had appeared within the quaternionic description of surfaces in $\mathbb{R}^3$ (formula (2.19) of [25]). But in [25] it was accompanied by another two equations (Eq. (2.16) of [25]) which are indispensable in the Sym's type approach. So the meaning of the system (2.5), seems, had been missing. The generalized Weierstrass type formulae admit also a beautiful formulation within the spinor representations of surfaces [26,27].

3. The Weierstrass representation for immersion into $\mathbb{R}^4$

An extension of the representation (2.5), (2.6) to the four-dimensional Euclidean space is as follows. Let ψ_1, φ_1 and ψ_2, φ_2 be two independent solutions of the system (2.5), i.e.

$$\psi_{1z} = p\varphi_1, \quad \psi_{2z} = p\varphi_2, \quad \varphi_{1\bar{z}} = -p\psi_1, \quad \varphi_{2\bar{z}} = -p\psi_2. \tag{3.1}$$

Eqs. (3.1) imply that

$$(\psi_1\psi_2)_z = -(\varphi_1\varphi_2)_{\bar{z}}, \quad (\psi_1\bar{\varphi}_2)_z = (\varphi_1\bar{\psi}_2)_{\bar{z}}. \tag{3.2}$$

As a consequence there are four functions $X^i(z,\bar{z})$ ($i = 1, 2, 3, 4$) such that

$$\begin{aligned} \mathrm{d}X^1 &= \frac{\mathrm{i}}{2}(\overline{\psi}_1\overline{\psi}_2 + \varphi_1\varphi_2)\,\mathrm{d}z + \text{c.c.}, \\ \mathrm{d}X^2 &= \tfrac{1}{2}(\overline{\psi}_1\overline{\psi}_2 - \varphi_1\varphi_2)\,\mathrm{d}z + \text{c.c}, \\ \mathrm{d}X^3 &= -\tfrac{1}{2}(\overline{\psi}_1\varphi_2 + \overline{\psi}_2\varphi_1)\,\mathrm{d}z + \text{c.c.}, \\ \mathrm{d}X^4 &= \mathrm{i}\tfrac{1}{2}(\overline{\psi}_1\varphi_2 - \overline{\psi}_2\varphi_1)\,\mathrm{d}z + \text{c.c.} \end{aligned} \tag{3.3}$$

where c.c. means complex conjugated previous term. We treat now these functions $X^i(z,\bar{z})$ as the coordinates of surface in $\mathbb{R}^4$. For components of induced metric

$$g_{zz} = \sum_{i=1}^{4}(X^i_z)^2 = \overline{g_{\bar{z}\bar{z}}}, \quad g_{z\bar{z}} = \sum_{i=1}^{4}(X^i_z X^i_{\bar{z}}) \tag{3.4}$$

one gets

$$g_{zz} = g_{\bar{z}\bar{z}} = 0 \tag{3.5}$$

and

$$g_{z\bar{z}} = \tfrac{1}{2}(|\psi_1|^2 + |\varphi_1|^2)(|\psi_2|^2 + |\varphi_2|^2). \tag{3.6}$$

Further, two normal vectors N_1, N_2 are

$$N_1 = \sqrt{\frac{|\varphi_1|^2|\varphi_2|^2}{u_1u_2}}\mathrm{Re}(A), \quad N_2 = \sqrt{\frac{|\varphi_1|^2|\varphi_2|^2}{u_1u_2}}\mathrm{Im}(A), \tag{3.7}$$

where

$$u_k = (|\psi_k|^2 + |\varphi_k|^2), \quad k = 1, 2, \tag{3.8}$$

$$A = \left[\mathrm{i}\left(\frac{\overline{\psi}_1}{\overline{\varphi}_1} - \frac{\overline{\psi}_2}{\varphi_2}\right), -\frac{\psi_1}{\varphi_1} + \frac{\overline{\psi}_2}{\varphi_2}, 1 - \frac{\psi_1\overline{\psi}_2}{\overline{\varphi}_1\varphi_2}, -\mathrm{i}\left(1 + \frac{\psi_1\overline{\psi}_2}{\overline{\varphi}_1\varphi_2}\right)\right]. \tag{3.9}$$

The mean curvature vector $\boldsymbol{H} = \boldsymbol{X}_{z\bar{z}}/g_{z\bar{z}}$ is given by

$$\begin{aligned} \boldsymbol{H} = \frac{2p}{u_1u_2}\mathrm{Re}[&-\mathrm{i}(\psi_1\varphi_2 + \psi_2\varphi_1), (\psi_1\varphi_2 + \psi_2\varphi_1), \\ &(\psi_1\overline{\psi}_2 - \varphi_1\overline{\varphi}_2), \mathrm{i}(\psi_1\overline{\psi}_2 - \varphi_1\overline{\varphi}_2)]. \end{aligned} \tag{3.10}$$

The components h_1, h_2 of $\vec{H}$ along N_1 and N_2 ($\boldsymbol{H} = h_1N_1 + h_2N_2$) are

$$h_1 = -p\frac{\varphi_1\overline{\varphi}_2 + \overline{\varphi}_1\varphi_2}{\sqrt{u_1u_2|\varphi_1|^2|\varphi_2|^2}}, \qquad h_2 = \mathrm{i}p\frac{\varphi_1\overline{\varphi}_2 - \overline{\varphi}_1\varphi_2}{\sqrt{u_1u_2|\varphi_1|^2|\varphi_2|^2}}. \tag{3.11}$$

So, the mean curvature $H = \sqrt{\sum_{i=1}^4 H^iH^i} = \sqrt{h_1^2 + h_2^2}$ is equal to

$$H = \frac{2p}{\sqrt{u_1u_2}}. \tag{3.12}$$

Then the Gaussian curvature is

$$K = -\frac{2}{u_1 u_2}[\log(u_1 u_2)]_{z\bar{z}}. \tag{3.13}$$

Finally, the Willmore functional $W = \int \boldsymbol{H}^2[\mathrm{d}S]$ is given by

$$W = 4\int p^2\,\mathrm{d}x\,\mathrm{d}y. \tag{3.14}$$

Thus, we have the following theorem.

Theorem 3.1. *The generalized Weierstrass formulae*

$$\begin{aligned}
X^1 &= \frac{\mathrm{i}}{2}\int_{\Gamma}[(\overline{\psi}_1\overline{\psi}_2 + \varphi_1\varphi_2)\,\mathrm{d}z' - (\psi_1\psi_2 + \overline{\varphi}_1\overline{\varphi}_2)\,\mathrm{d}\bar{z}'],\\
X^2 &= \frac{1}{2}\int_{\Gamma}[(\overline{\psi}_1\overline{\psi}_2 - \varphi_1\varphi_2)\,\mathrm{d}z' + (\psi_1\psi_2 - \overline{\varphi}_1\overline{\varphi}_2)\,\mathrm{d}\bar{z}'],\\
X^3 &= -\frac{1}{2}\int_{\Gamma}[(\overline{\psi}_1\varphi_2 + \overline{\psi}_2\varphi_1)\,\mathrm{d}z' + (\psi_1\overline{\varphi}_2 + \psi_2\overline{\varphi}_1)\,\mathrm{d}\bar{z}'],\\
X^4 &= \frac{i}{2}\int_{\Gamma}[(\overline{\psi}_1\varphi_2 - \overline{\psi}_2\varphi_1)\,\mathrm{d}z' - (\psi_1\overline{\varphi}_2 - \psi_2\overline{\varphi}_1)\,\mathrm{d}\bar{z}'],
\end{aligned} \tag{3.15}$$

where

$$\psi_{\alpha z} = p\varphi_\alpha, \quad \varphi_{\alpha\bar{z}} = -p\psi_\alpha, \qquad \alpha = 1, 2. \tag{3.16}$$

$p(z, \bar{z})$ is a real valued function, Γ is a contour in $\mathbb{C}$, define the conformal immersion of a surface into $\mathbb{R}^4$. The induced metric is of the form

$$\mathrm{d}s^2 = u_1 u_2\,\mathrm{d}z\,\mathrm{d}\bar{z}, \tag{3.17}$$

where $u_\alpha = |\psi_\alpha|^2 + |\varphi_\alpha|^2$ ($\alpha = 1, 2$), the Gaussian and mean curvatures are

$$K = -\frac{2}{u_1 u_2}[\log(u_1 u_2)]_{z\bar{z}}, \quad H = \frac{2p}{\sqrt{u_1 u_2}}. \tag{3.18}$$

The total squared mean curvature (Willmore functional) is given by

$$W = 4\int p^2\,\mathrm{d}x\,\mathrm{d}y. \tag{3.19}$$

The generalized Weierstrass representation (3.15) defines surface in $\mathbb{R}^4$ up to translations. In the particular case $\psi_2 = \pm\psi_1$, $\varphi_2 = \pm\varphi_1$, $X^4_z = X^4_{\bar{z}} = 0$ and formulae (3.16)–(3.19) are reduced to those (2.5)–(2.9) of $\mathbb{R}^3$ case.

Corollary 3.1. *Minimal surfaces in $\mathbb{R}^4$ are given by the Weierstrass representation* (3.15), (3.16) *with* $p(z,\bar{z}) = 0$. *Surfaces of constant mean curvature H are given by formulae* (3.15) *where* $\psi_\alpha, \varphi_\alpha$ $(\alpha = 1, 2)$ *obey the system of equations*

$$\begin{aligned}\psi_{\alpha z} &= \frac{H}{2}\sqrt{(|\psi_1|^2 + |\varphi_1|^2)(|\psi_2|^2 + |\varphi_2|^2)}\varphi_\alpha,\\ \varphi_{\alpha\bar{z}} &= -\frac{H}{2}\sqrt{(|\psi_1|^2 + |\varphi_1|^2)(|\psi_2|^2 + |\varphi_2|^2)}\psi_\alpha.\end{aligned} \tag{3.20}$$

At $\psi_2 = \pm\psi_1$, $\varphi_2 = \pm\varphi_1$ the system (3.20) is reduced to a simpler one [11] which has been studied in [16].

Note Eqs. (3.3) can be represented in the form

$$\begin{aligned}\mathrm{d}(X^1 + \mathrm{i}X^2) &= \mathrm{i}\bar{\psi}_1\bar{\psi}_2\,\mathrm{d}z - \mathrm{i}\bar{\varphi}_1\bar{\varphi}_2\,\mathrm{d}\bar{z},\\ \mathrm{d}(X^1 - \mathrm{i}X^2) &= \mathrm{i}\varphi_1\varphi_2\,\mathrm{d}z - \mathrm{i}\psi_1\psi_2\,\mathrm{d}\bar{z},\\ \mathrm{d}(X^4 + \mathrm{i}X^3) &= -\mathrm{i}\bar{\psi}_2\varphi_1\,\mathrm{d}z - \mathrm{i}\psi_1\bar{\varphi}_2\,\mathrm{d}\bar{z},\\ \mathrm{d}(X^4 - \mathrm{i}X^3) &= \mathrm{i}\bar{\psi}_1\varphi_2\,\mathrm{d}z + \mathrm{i}\psi_2\bar{\varphi}_1\,\mathrm{d}\bar{z},\end{aligned} \tag{3.21}$$

which reveals a symmetry between the pairs of coordinates (X^1, X^2), (X^3, X^4).

Formulae (3.3) can be also rewritten in a spinor representation type form

$$\mathrm{d}(\sigma_1 X^2 + \sigma_2 X^1 - \sigma_3 X^3 + \mathrm{i}IX^4) = V_2^\dagger \begin{pmatrix} 0 & \mathrm{d}z \\ \mathrm{d}\bar{z} & 0 \end{pmatrix} V_1, \tag{3.22}$$

where

$$V_{1,2} = \begin{pmatrix} \psi_{(1,2)} & -\bar{\varphi}_{(1,2)} \\ \varphi_{(1,2)} & \bar{\psi}_{(1,2)} \end{pmatrix}, \tag{3.23}$$

σ_i $(i = 1, 2, 3)$ are the Pauli matrices and I is the identity matrix.

Condition (3.5), that an immersion is conformal, written as

$$(X^1_z)^2 + (X^2_z)^2 + (X^3_z)^2 + (X^4_z)^2 = 0 \tag{3.24}$$

defines the complex quadric Q_2

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0 \tag{3.25}$$

in CP^3 where w_i $(i = 1, 2, 3, 4)$ are homogeneous coordinates. A diffeomorphism of Q_2 to the Grassmannian $G_{2,4}$ of oriented 2-planes in $\mathbb{R}^4$ allows us to define the Gauss map $G(z)$ for a surface represented by the generalized Weierstrass formulae (3.15). It is given by

$$\boldsymbol{G}(z) = [\mathrm{i}(\bar{\psi}_1\bar{\psi}_2 + \varphi_1\varphi_2), \bar{\psi}_1\bar{\psi}_2 - \varphi_1\varphi_2, -\bar{\psi}_1\varphi_2 - \bar{\psi}_2\varphi_1, \mathrm{i}(\bar{\psi}_1\varphi_2 - \bar{\psi}_2\varphi_1)]. \tag{3.26}$$

The Gauss map for surfaces immersed into $\mathbb{R}^4$ has been studied earlier in the paper [13]. In [13] the Gauss map $\boldsymbol{G}(z)$ has been parametrized as follows:

$$G(z) = [(1 + f_1 f_2), \mathrm{i}(1 - f_1 f_2), (f_1 - f_2), -\mathrm{i}(f_1 + f_2)], \tag{3.27}$$

where f_1 and f_2 are complex-valued functions. A surface in $\mathbb{R}^4$ is then defined by [13]

$$X = \int^{z} \mathrm{Re}(\eta G\,\mathrm{d}z), \tag{3.28}$$

where f_1 and f_2 satisfy the compatibility conditions

$$\mathrm{Im}\left[\left(\frac{f_{1z\bar{z}}}{f_{1\bar{z}}} - \frac{2\bar{f}_1 f_{1z}}{1 + |f_1|^2}\right)_{\bar{z}} + \left(\frac{f_{2z\bar{z}}}{f_{2\bar{z}}} - \frac{2\bar{f}_2 f_{2z}}{1 + |f_2|^2}\right)_{\bar{z}}\right] = 0 \tag{3.29}$$

and

$$|F_1| = |F_2| \tag{3.30}$$

where $F_i = f_{i\bar{z}}(1 + |f_i|^2)^{-1}, i = 1, 2$. The function η is given by

$$\bar{\eta}^2 = -\frac{4F_1 F_2}{H^2(1 + |f_1|^2)(1 + |f_2|^2)}, \tag{3.31}$$

where the mean curvature H is expressed via f_1 and f_2 by

$$2(\log H)_z = \frac{f_{1z\bar{z}}}{f_{1\bar{z}}} - \frac{2\bar{f}_1 f_{1z}}{1 + |f_1|^2} + \frac{f_{2z\bar{z}}}{f_{2\bar{z}}} - \frac{2\bar{f}_2 f_{2z}}{1 + |f_2|^2}. \tag{3.32}$$

Similar to the three-dimensional case this representation includes the complicated compatibility conditions.

Theorem 3.2. *The generalized Weierstrass representation* (3.15)–(3.19) *implies the Gauss map type representation* (3.28)–(3.32) *via the substitution*

$$\eta = \mathrm{i}\varphi_1\varphi_2, \quad f_1 = \mathrm{i}\frac{\bar{\psi}_1}{\varphi_1}, \quad f_2 = -\mathrm{i}\frac{\bar{\psi}_2}{\varphi_2}. \tag{3.33}$$

The proof is straightforward: Eqs. (3.16) and (3.33) give the constraints (3.29)–(3.31) with

$$p = -\mathrm{i}F_1\frac{\varphi_1}{\bar{\varphi}_1} = \mathrm{i}F_2\frac{\varphi_2}{\bar{\varphi}_2} \tag{3.34}$$

while (3.15) is converted into (3.28).

4. Surfaces in four-dimensional Riemann space

The results of the previous section apparently can be extended to the case of immersion into generic four-dimensional Riemann space with the metric tensor g_{ik}.

Proposition 4.1. *Formulae* (3.15), (3.16) *define an immersion of surface into the four-dimensional Riemann space with the metric tensor g_{ik}. The induced metric is*

$$ds^2 = g_{zz}\,dz^2 + 2g_{z\bar{z}}\,dz\,d\bar{z} + g_{\bar{z}\bar{z}}\,d\bar{z}^2, \tag{4.1}$$

where

$$g_{zz} = g_{ik}X^i_z X^k_z = \overline{g_{\bar{z}\bar{z}}}, \quad g_{z\bar{z}} = g_{ik}X^i_z X^k_{\bar{z}}. \tag{4.2}$$

The Gaussian and mean curvature are calculated straightforwardly.

In the case of conformally Euclidean spaces $g_{ik} = e^{2\sigma}\delta_{ik}$ ($i, k = 1, 2, 3, 4, \sigma$ is a function and δ_{ik} is the Kronecker symbol) the immersion is the conformal one:

$$ds^2 = e^{2\sigma} u_1 u_2\, dz\, d\bar{z}. \tag{4.3}$$

The Gaussian and mean curvatures are

$$K = -2e^{-2\sigma}\frac{[2\sigma + \log(u_1 u_2)]_{z\bar{z}}}{u_1 u_2}, \quad H = 2e^{-\sigma}\frac{p}{\sqrt{u_1 u_2}}. \tag{4.4}$$

For the Willmore functional one gets

$$W = 4\int p^2\, dx\, dy. \tag{4.5}$$

A special case of immersions into the space $\mathbb{S}^4$ of constant curvature had attracted recently the particular interest (see e.g. [22,28]). To describe it we choose the Riemann form for the metric of $\mathbb{S}^4$, i.e. (see e.g. [29])

$$e^{\sigma} = \left[1 + \frac{K_0}{4}\sum_{i=1}^{4}(X^i)^2\right]^{-1}, \tag{4.6}$$

where K_0 is the curvature. Then formulae (3.15), (3.16), (4.3)–(4.5) define the conformal immersion of a surface into $\mathbb{S}^4$. At $\psi_2 = \pm\psi_1, \varphi_2 = \pm\varphi_1$ ($X^4 = 0$) one has the conformal immersion into $\mathbb{S}^3$. The generalized Weierstrass representation provides us an effective method to study immersions into $\mathbb{S}^3$ and $\mathbb{S}^4$, in particular the Willmore surfaces. It will be done in a separate paper.

Immersions into the Riemann spaces with non-Euclidean signature are of interest too. For Minkowski space M^4: $g_{ik} = \mathrm{diag}(1, 1, 1 - 1)$. Formulae (3.15), (3.16), (4.1), (4.2) define a surface in M^4 with the line element

$$ds^2 = [2g_{z\bar{z}} + 2\mathrm{Re}(g_{zz})]\,dx^2 - 4\mathrm{Im}(g_{zz})\,dx\,dy + [2g_{z\bar{z}} - 2\mathrm{Re}(g_{zz})]\,dy^2, \tag{4.7}$$

where $z = x + iy$ and

$$g_{zz} = \tfrac{1}{2}(\bar{\psi}_1\varphi_2 - \bar{\psi}_2\varphi_1)^2 = \overline{g_{\bar{z}\bar{z}}}, \quad g_{z\bar{z}} = \tfrac{1}{2}|\bar{\psi}_1\varphi_2 + \bar{\psi}_2\varphi_1|^2. \tag{4.8}$$

For a space with the metric $g_{ik} = \mathrm{diag}(1, 1, -1, -1)$, in addition to the immersion of the type (4.7), (4.8) there is a Weierstrass type representation for conformal immersion.

Theorem 4.1. *The Weierstrass type formulae*

$$
\begin{aligned}
X^1 &= \frac{i}{2}\int_\Gamma [(\overline{\psi}_1\overline{\psi}_2 - \varphi_1\varphi_2)\,dz' - (\psi_1\psi_2 - \overline{\varphi}_1\overline{\varphi}_2)\,d\bar{z}'],\\
X^2 &= \frac{1}{2}\int_\Gamma [(\overline{\psi}_1\overline{\psi}_2 + \varphi_1\varphi_2)\,dz' + (\psi_1\psi_2 + \overline{\varphi}_1\overline{\varphi}_2)\,d\bar{z}'],\\
X^3 &= -\frac{1}{2}\int_\Gamma [(\overline{\psi}_1\varphi_2 + \overline{\psi}_2\varphi_1)\,dz' + (\psi_1\overline{\varphi}_2 + \psi_2\overline{\varphi}_1)\,d\bar{z}'],\\
X^4 &= \frac{i}{2}\int_\Gamma [(\overline{\psi}_1\varphi_2 - \overline{\psi}_2\varphi_1)\,dz' - (\psi_1\overline{\varphi}_2 - \psi_2\overline{\varphi}_1)\,d\bar{z}'],
\end{aligned}
\tag{4.9}
$$

where

$$\psi_{\alpha z} = p\varphi_\alpha, \quad \varphi_{\alpha\bar{z}} = p\psi_\alpha, \qquad \alpha = 1, 2, \tag{4.10}$$

and p is a real-valued function, define a conformal immersion of surface into the four-dimensional space with the metric $g_{ik} = \mathrm{diag}(1, 1, -1, -1)$. The induced metric is of the form

$$ds^2 = (|\psi_1|^2 - |\varphi_1|^2)(|\psi_2|^2 - |\varphi_2|^2)\,dz\,d\bar{z} = v\,dz\,d\bar{z} \tag{4.11}$$

and the Gaussian and mean curvatures are

$$K = -\frac{2}{v}(\log v)_{z\bar{z}}, \quad \boldsymbol{H}^2 = -\frac{4p^2}{v}. \tag{4.12}$$

The total squared mean curvature is

$$W = \int \boldsymbol{H}^2[dS] = -4\int p^2\,dx\,dy. \tag{4.13}$$

The proof is similar to that of Theorem 3.1. In this case

$$(\psi_1\psi_2)_z = (\varphi_1\varphi_2)_{\bar{z}}, \quad (\psi_1\overline{\varphi}_2)_z = (\overline{\psi}_2\varphi_1)_{\bar{z}} \tag{4.14}$$

and

$$
\begin{aligned}
d(X^1 + iX^2) &= i\overline{\psi}_1\overline{\psi}_2\,dz + i\overline{\varphi}_1\overline{\varphi}_2\,d\bar{z},\\
d(X^1 - iX^2) &= -i\varphi_1\varphi_2\,dz - i\psi_1\psi_2\,d\bar{z},\\
d(X^4 + iX^3) &= -i\overline{\psi}_2\varphi_1\,dz - i\psi_1\overline{\varphi}_2\,d\bar{z},\\
d(X^4 - iX^3) &= i\overline{\psi}_1\varphi_2\,dz + i\psi_2\overline{\varphi}_1\,d\bar{z}.
\end{aligned}
\tag{4.15}
$$

The Weierstrass representation (4.9)–(4.13) could be useful also for the study of $N = 2$ superstring [30].

5. Surfaces in multi-dimensional spaces and on Lie groups

Any solution (ψ, φ) of the system (2.5) gives rise via (2.6) to the three coordinates X^1, X^2, X^3. Given a pair of solutions (ψ_1, φ_1) and (ψ_2, φ_2) one has two possibilities. The first is to generate four coordinates via the formula (3.15), the second is to get six coordinates: X^1, X^2, X^3 via (2.6) with the solutions (ψ_1, φ_1) and X^4, X^5, X^6 via (2.6) using the solutions (ψ_2, φ_2). In the latter case, one has the conformal immersion of a surface into $\mathbb{R}^6$ with the induced metric

$$\mathrm{d}s^2 = (u_1^2 + u_2^2)\,\mathrm{d}z\,\mathrm{d}\bar{z}. \tag{5.1}$$

Further, introducing four coordinates X^7, X^8, X^9, X^{10} via (3.15) one can get an immersion into $\mathbb{R}^{10}$. The induced metric in this case is

$$\mathrm{d}s^2 = (u_1^2 + u_1u_2 + u_2^2)\,\mathrm{d}z\,\mathrm{d}\bar{z}. \tag{5.2}$$

In such a manner apparently one can get an immersion of a surface into the Euclidean space $\mathbb{R}^N$ with $N = 3n + 4m$ where n and m are arbitrary integers. The corresponding induced metric is of the form

$$\mathrm{d}s^2 = \left(\sum_{\alpha=1}^{n} u_\alpha^2 + \sum_{\alpha\neq\beta}^{m} u_\alpha u_\beta\right)\mathrm{d}z\,\mathrm{d}\bar{z}, \tag{5.3}$$

where m is equal to the number of pairs α, β $(\alpha \neq \beta)$.

The Weierstrass representations for immersion of surfaces into the complex spaces are defined analogously. Let $(\psi_\alpha, \varphi_\alpha)$ and $(\psi_\beta, \varphi_\beta)$ be any two solutions of the system (2.5), i.e.

$$\psi_{\alpha z} = p\varphi_\alpha, \quad \psi_{\beta z} = p\varphi_\beta, \quad \varphi_{\alpha\bar{z}} = -p\psi_\alpha, \quad \varphi_{\beta\bar{z}} = -p\psi_\beta, \tag{5.4}$$

where p is a complex valued function. The system (5.4) implies

$$(\psi_\alpha\psi_\beta)_z = -(\varphi_\alpha\varphi_\beta)_{\bar{z}}. \tag{5.5}$$

Consequently, the integral

$$X^\gamma = \sum_{\alpha,\beta} A^\gamma_{\alpha\beta} \int_\Gamma (\psi_\alpha\psi_\beta\,\mathrm{d}\bar{z} - \varphi_\alpha\varphi_\beta\,\mathrm{d}z), \tag{5.6}$$

where $A^\gamma_{\alpha\beta}$ are arbitrary constants does not depend on the contour of integration Γ. Treating N functions $X^\gamma(z, \bar{z})$ $(\gamma = 1, \ldots, N)$ as the coordinates in $\mathbb{C}^N$, one gets an immersion of a surface into $\mathbb{C}^N$. The induced metric is given by

$$g_{zz} = \left(\sum_{\gamma=1}^{N}\sum_{\alpha\beta} A^\gamma_{\alpha\beta}\psi_\alpha\psi_\beta\right)^2.$$

$$g_{z\bar{z}} = -\left(\sum_{\gamma=1}^{N}\sum_{\alpha\beta} A^{\gamma}_{\alpha\beta}\psi_\alpha\psi_\beta\right)\left(\sum_{\gamma=1}^{N}\sum_{\alpha\beta} A^{\gamma}_{\alpha\beta}\varphi_\alpha\varphi_\beta\right), \tag{5.7}$$

$$g_{\bar{z}\bar{z}} = \left(\sum_{\gamma=1}^{N}\sum_{\alpha\beta} A^{\gamma}_{\alpha\beta}\varphi_\alpha\varphi_\beta\right)^2 .$$

Choosing $A^{\gamma}_{\alpha\beta}$ properly, one can get a conformal immersion.

Finally, let us consider a set of solutions $\psi^{(i)}_\alpha$, $\varphi^{(i)}_\alpha$ which solve the systems

$$\psi^{(i)}_{\alpha z} = p^{(i)}\varphi^{(i)}_\alpha, \quad i = 1, \dots, N, \quad \varphi^{(i)}_{\alpha\bar{z}} = -p^{(i)}\psi^{(i)}_\alpha, \qquad \alpha = 1, \dots, M, \tag{5.8}$$

with different potentials $p^{(i)}$. Since

$$[\psi^{(i)}_\alpha\psi^{(i)}_\beta]_z = -[\varphi^{(i)}_\alpha\varphi^{(i)}_\beta]_{\bar{z}}, \tag{5.9}$$

one can define a set of functions $X^{\alpha\beta}$ via

$$X^{\alpha\beta} = \sum_{i=1}^{N} B_i \int_{\Gamma}^{z} [\psi^{(i)}_\alpha\psi^{(i)}_\beta \, \mathrm{d}\bar{z} - \varphi^{(i)}_\alpha\varphi^{(i)}_\beta \, \mathrm{d}z], \tag{5.10}$$

where B_i are arbitrary constants. Formula (5.10) defines, in fact, a matrix $\mathbf{X}$ such that $\mathbf{X}_{\alpha\beta} = X^{\alpha\beta}$ $(\alpha, \beta = 1, \dots, M)$. At $N \geq M$ the matrix X is a generic element of the group $GL(M, C)$. So formula (5.10) with $N = M$ defines a surface on the group $GL(M, C)$ in a meaning given in [31]. Using the formulae from [31], one can calculate all characteristics of such surfaces.

6. Integrable deformations

An important advantage of the generalized Weierstrass representation is that it provides a way to construct integrable deformations of immersed surfaces. The idea is the following [10,11]: let the functions p, ψ and φ in (2.5) depend on the parameter t. Then one considers those deformations of ψ and φ that there are differential operators A, B, C, D such that

$$\psi_t = A\psi + B\varphi, \quad \varphi_t = C\psi + D\varphi. \tag{6.1}$$

Given A, B, C, D the compatibility condition of (6.1) with (2.5) is equivalent to the nonlinear partial differential equation for p. Varying operators A, B, C, D, one gets an infinite hierarchy of integrable equations for p. It is the so-called modified Veselov–Novikov (mVN) hierarchy [11]. Choosing A, B, C, D, as the first-order operators, one gets the linear equation for p. The first non-trivial nonlinear example is given by the modified Veselov–Novikov equation:

$$p_t = p_{zzz} + 3p_z\omega + \tfrac{3}{2}p\omega_z + \text{c.c.}, \qquad \omega_{\bar{z}} = (p^2)_z. \tag{6.2}$$

The corresponding operators are

$$A = \partial_z^3 + \partial_{\bar z}^3 + 3\overline{\omega}\partial_{\bar z} + \tfrac{3}{2}\overline{\omega}_{\bar z}, \tag{6.3}$$

$$B = -3p_z\partial_z + 3p\omega, \tag{6.4}$$

$$C = 3p_{\bar z}\partial_{\bar z} - 3p\overline{\omega}, \tag{6.5}$$

$$D = \partial_z^3 + \partial_{\bar z}^3 + 3\omega\partial_z + \tfrac{3}{2}\omega_z. \tag{6.6}$$

The deformation of ψ, φ via (6.1) generates the corresponding deformations of the coordinates $X^i(z, \bar z, t)$. In the case of immersion into $\mathbb{R}^3$ it was shown in [14,15] that the Willmore functional W is invariant under the deformations generated by the mVN equation (6.2) as well as by the whole mVN hierarchy. Formulae for deformation of coordinates, elements of metrics and other geometric quantities have been obtained in [19].

To get integrable deformations of surfaces immersed in $\mathbb{R}^4$ we assume that both solutions (ψ_1, φ_1) and (ψ_2, φ_2) of the system (3.16) evolve in t according to Eq. (6.1) with the same A, B, C, D. Correspondingly the coordinates X^i $(i = 1, 2, 3, 4)$ of the surface given by (3.15) are deformed in t. These deformations of a surface are integrable one similar to the case $\mathbb{R}^3$ [11]. From (3.19) and equality $\int (p^2)_t \, \mathrm{d}z \, \mathrm{d}\bar z = 0$ it immediately follows:

Theorem 6.1. *The value of the Willmore functional W for surface immersed into $\mathbb{R}^4$ is preserved by the mVN deformations (by all hierarchy).*

There is an infinite set of functionals over p preserved by the evolution (6.2). So there is an infinite family of geometric functionals over surface in $\mathbb{R}^4$ which are preserved by the mVN deformations. The Willmore functional W for surfaces in $\mathbb{R}^4$ is invariant under conformal transformations in this space (see e.g. [22]). One could conjecture that similar to the $\mathbb{R}^3$ case [18] all these higher preserved functionals are invariant under the conformal transformations in $\mathbb{R}^4$ too.

Formulae (6.1), (6.2) define also integrable deformations of surfaces in Riemann spaces considered above. In particular we have the following theorem.

Theorem 6.2. *The value of Willmore functional for surfaces immersed into conformally-Euclidean spaces is preserved by the mVN deformations.*

Using the results of the papers [17,20,21], one can define the global deformations of surfaces immersed into $\mathbb{R}^4$, $\mathbb{S}^4$ and other spaces. This problem will be considered elsewhere.

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